

P-CONNECTION ON RIEMANNIAN ALMOST PRODUCT MANIFOLDS

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Abstract

In the present work¹, we introduce a linear connection (preserving the almost product structure and the Riemannian metric) on Riemannian almost product manifolds. This connection, called P -connection, is an analogue of the first canonical connection of Lichnerowicz in the Hermitian geometry and the B -connection in the geometry of the almost complex manifolds with Norden metric. Particularly, we consider the P -connection on a the class of manifolds with nonintegrable almost product structure.

Key words: Riemannian manifold, Riemannian metric, almost product structure, linear connection, parallel torsion.

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1. INTRODUCTION

In [1] a linear connection, called B -connection, is introduced on almost complex manifolds with Norden (or anti-Hermitian) metric. This connection (preserving the almost complex structure and the Norden metric) is an analogue of the first canonical connection of Lichnerowicz in the Hermitian geometry ([2], [3], [4]). In [5] the B -connection is considered on a class of almost complex manifolds with Norden metric and nonintegrable almost complex structure. This is the class \mathcal{W}_3 of the quasi-Kähler manifolds with Norden metric.

In the present work, we introduce a linear connection (preserving the almost product structure and the Riemannian metric) on Riemannian almost product manifolds. This connection, called P -connection, is an analogue of the first canonical connection of Lichnerowicz in the Hermitian geometry and the B -connection in the geometry of the almost complex manifolds with Norden metric. Particularly, we consider the P -connection on the manifolds of the class \mathcal{W}_3 from the classification in [6].

The systematic development of the theory of Riemannian almost product manifolds was started by K. Yano [4]. In [7] A. M. Naveira gives a classification of these manifolds with respect to the covariant differentiation of the almost product structure. Having in mind the results in [7], M. Staikova and

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K. Gribachev give in [6] a classification of the Riemannian almost product manifolds with zero trace of the almost product structure.

2. PRELIMINARIES

Let (M, P, g) be a *Riemannian almost product manifold*, i.e. a differentiable manifold M with a tensor field P of type $(1, 1)$ and a Riemannian metric g such that

$$(2.1) \quad P^2x = x, \quad g(Px, Py) = g(x, y)$$

for arbitrary x, y of the algebra $\mathfrak{X}(M)$ of the smooth vector fields on M . Obviously $g(Px, y) = g(x, Py)$.

Further x, y, z, w will stand for arbitrary elements of $\mathfrak{X}(M)$.

In this work we consider Riemannian almost product manifolds with $\text{tr}P = 0$. In this case (M, P, g) is an even-dimensional manifold.

The classification in [6] of Riemannian almost product manifolds is made with respect to the tensor field F of type $(0, 3)$, defined by

$$(2.2) \quad F(x, y, z) = g((\nabla_x P)y, z),$$

where ∇ is the Levi-Civita connection of g . The tensor F has the following properties:

$$(2.3) \quad \begin{aligned} F(x, y, z) &= F(x, z, y) = -F(x, Py, Pz), \\ F(x, y, Pz) &= -F(x, Py, z). \end{aligned}$$

The basic classes of the classification in [6] are \mathcal{W}_1 , \mathcal{W}_2 and \mathcal{W}_3 . Their intersection is the class \mathcal{W}_0 of the *Riemannian P -manifolds*, determined by the condition $F(x, y, z) = 0$ or equivalently $\nabla P = 0$. In the classification there are include the classes $\mathcal{W}_1 \oplus \mathcal{W}_2$, $\mathcal{W}_1 \oplus \mathcal{W}_3$, $\mathcal{W}_2 \oplus \mathcal{W}_3$ and the class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ of all Riemannian almost product manifolds.

In the present work we consider manifolds from the class \mathcal{W}_3 . This class is determined by the condition

$$(2.4) \quad \mathfrak{S}_{x,y,z} F(x, y, z) = 0,$$

where $\mathfrak{S}_{x,y,z}$ is the cyclic sum by x, y, z . This is the only class of the basic classes \mathcal{W}_1 , \mathcal{W}_2 and \mathcal{W}_3 , where each manifold (which is not Riemannian P -manifold) has a nonintegrable almost product structure P . This means that in \mathcal{W}_3 the Nijenhuis tensor N , determined by

$$N(x, y) = (\nabla_x P)Py - (\nabla_{Px})Py + (\nabla_y P)Px - (\nabla_{Py})Px,$$

is non-zero.

Further, manifolds of the class \mathcal{W}_3 we call *Riemannian \mathcal{W}_3 -manifolds*.

As it is known the curvature tensor field R of a Riemannian manifold with metric g is determined by $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]}z$ and the

corresponding tensor field of type $(0, 4)$ is defined as follows $R(x, y, z, w) = g(R(x, y)z, w)$.

Let (M, P, g) be a Riemannian almost product manifold and $\{e_i\}$ be a basis of the tangent space $T_p M$ at a point $p \in M$. Let the components of the inverse matrix of g with respect to $\{e_i\}$ be g^{ij} . If ρ and τ are the Ricci tensor and the scalar curvature, then ρ^* and τ^* , defined by $\rho^*(y, z) = g^{ij}R(e_i, y, z, Pe_j)$ and $\tau^* = g^{ij}\rho^*(e_i, e_j)$, are called an *associated Ricci tensor* and an *associated scalar curvature*, respectively. We will use also the trace $\tau^{**} = g^{ij}g^{ks}R(e_i, e_k, Pe_s, Pe_j)$.

The *square norm* of ∇P is defined by

$$(2.5) \quad \|\nabla P\|^2 = g^{ij}g^{ks}g((\nabla_{e_i}P)e_k, (\nabla_{e_j}P)e_s).$$

Obviously $\|\nabla P\|^2 = 0$ iff (M, P, g) is a Riemannian P -manifold. In [8] it is proved that if (M, P, g) is a Riemannian \mathcal{W}_3 -manifold then

$$(2.6) \quad \|\nabla P\|^2 = -2g^{ij}g^{ks}g((\nabla_{e_i}P)e_k, (\nabla_{e_s}P)e_j) = 2(\tau - \tau^{**}).$$

A tensor L of type $(0, 4)$ with the properties

$$(2.7) \quad L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),$$

$$(2.8) \quad \sum_{x,y,z} L(x, y, z, w) = 0 \quad (\text{the first Bianchi identity})$$

is called a *curvature-like tensor*. Moreover, if the curvature-like tensor L has the property

$$(2.9) \quad L(x, y, Pz, Pw) = L(x, y, z, w),$$

we call it a *Riemannian P -tensor*.

If the curvature tensor R on a Riemannian \mathcal{W}_3 -manifold (M, P, g) is a Riemannian P -tensor, i.e. $R(x, y, Pz, Pw) = R(x, y, z, w)$, then $\tau^{**} = \tau$. Therefore $\|\nabla P\|^2 = 0$, i.e. (M, P, g) is a Riemannian P -manifold.

3. P-CONNECTION

A linear connection ∇' on a Riemannian almost product manifold (M, P, g) preserving P and g , i.e. $\nabla'P = \nabla'g = 0$, is called a *natural connection* [9].

Definition 3.1. *The natural connection ∇' on a Riemannian almost product manifold (M, P, g) determined by*

$$(3.1) \quad \nabla'_x y = \nabla_x y - \frac{1}{2}(\nabla_x P)y,$$

is called a P -connection.

Let T be a torsion tensor of the P -connection ∇' determined on (M, P, g) by (3.1). Because of the symmetry of ∇ , from (3.1) we have $T(x, y) = -\frac{1}{2}\{(\nabla_x P)Py - (\nabla_y P)Px\}$. Then, having in mind (3.1), we obtain

$$T(x, y, z) = g(T(x, y), z) = -\frac{1}{2}\{F(x, Py, z) - F(y, Px, z)\}.$$

Hence and (2.3) we have

$$(3.2) \quad \underset{x, y, z}{\mathfrak{S}} T(x, y, Pz) = 0.$$

Let Q be the tensor field determined by

$$(3.3) \quad Q(y, z) = -\frac{1}{2}(\nabla_y P)Pz.$$

Having in mind (2.2), for the corresponding (0,3)-tensor field we have

$$(3.4) \quad Q(y, z, w) = -\frac{1}{2}F(y, Pz, w).$$

Because of the properties (2.3), (3.4) implies $Q(y, z, w) = -Q(y, w, z)$.

Let R' be the curvature tensor of the P -connection ∇' . Then, according to (3.1) and (2.5) we have [10]

$$(3.5) \quad \begin{aligned} R'(x, y, z, w) &= R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) \\ &\quad + Q(x, Q(y, z), w) - Q(y, Q(x, z), w). \end{aligned}$$

After a covariant differentiation of (3.4), a substitution in (3.5), a use of (2.2), (2.3), (2.4) and some calculations, from (3.5) we obtain

$$\begin{aligned} R'(x, y, z, w) &= R(x, y, z, w) + \frac{1}{2}[(\nabla_x F)(y, z, Pw) - (\nabla_y F)(x, z, Pw)] \\ &\quad + \frac{1}{4}[g((\nabla_y P)z, (\nabla_x P)w) - g((\nabla_x P)z, (\nabla_y P)w)]. \end{aligned}$$

The last equality, having in mind the Ricci identity for Riemannian almost product manifolds

$$(\nabla_x F)(y, z, w) - (\nabla_y F)(x, z, w) = R(x, y, Pz, w) - R(x, y, z, Pw),$$

implies

$$(3.6) \quad \begin{aligned} R'(x, y, z, w) &= \frac{1}{4}\{2R(x, y, z, w) + 2R(x, y, Pz, Pw) + K(x, y, z, w)\}, \end{aligned}$$

where K is the tensor determined by

$$(3.7) \quad K(x, y, z, w) = -g((\nabla_x P)z, (\nabla_y P)w) + g((\nabla_y P)z, (\nabla_x P)w).$$

In this way, the following theorem is valid.

Theorem 3.1. *The curvature tensor R' of the P -connection ∇' on a Riemannian almost product manifold (M, P, g) has the form (3.6). \square*

From (3.6) it follows immediately that the property (2.7) and (2.9) are valid for R' . Therefore, the property (2.8) for R' is a necessary and sufficient condition for R' to be a Riemannian P -tensor. Since R satisfies (2.8), then from (3.6) we obtain immediately the following

Theorem 3.2. *The curvature tensor R' of the P -connection ∇' on a Riemannian \mathcal{W}_3 -manifold (M, P, g) is a Riemannian P -tensor iff*

$$(3.8) \quad 2 \underset{x,y,z}{\mathfrak{S}} R(x, y, Pz, Pw) = - \underset{x,y,z}{\mathfrak{S}} K(x, y, z, w).$$

□

Let the following condition be valid for the Riemannian almost product manifold (M, P, g) :

$$(3.9) \quad \underset{x,y,z}{\mathfrak{S}} R(x, y, Pz, Pw) = 0.$$

We say that the condition (3.9) characterizes a class \mathcal{L}_2 of the Riemannian almost product manifolds.

The equality (3.7) implies immediately the properties (2.7) and (2.9) for P . Then, according to (3.8) and (3.9), we obtain the following

Theorem 3.3. *Let (M, P, g) belong to the class \mathcal{L}_2 . Then the curvature tensor R' of the P -connection ∇' is a Riemannian P -tensor iff the tensor P determined by (3.7) is a Riemannian P -tensor, too.*

□

Having in mind (3.6), the last theorem implies the following

Corollary 3.4. *Let the curvature tensor R' of the P -connection ∇' be a Riemannian P -tensor on $(M, P, g) \in \mathcal{L}_2$. Then the tensor H , determined by*

$$(3.10) \quad H(x, y, z, w) = R(x, y, z, w) + R(x, y, Pz, Pw)$$

is a Riemannian P -tensor, too.

□

4. CURVATURE PROPERTIES OF THE P -CONNECTION IN $\mathcal{W}_3 \cap \mathcal{L}_2$

Let us consider the manifold $(M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_2$ with a Riemannian P -tensor of curvature R' of the P -connection ∇' . Then, according to Theorem 3.3 and Corollary 3.4, the tensors K and H , determined by (3.7) and (3.10), respectively, are also Riemannian P -tensors.

Let ρ' and $\rho(K)$ be the Ricci tensors for R' and K , respectively. Then we obtain immediately from (3.6)

$$(4.1) \quad \rho(y, z) + \rho^*(y, Pz) = 2\rho'(y, z) - \frac{1}{2}\rho(K)(y, z).$$

From (4.1) we have

$$(4.2) \quad \tau + \tau^{**} = 2\tau' - \frac{1}{2}\tau(K),$$

where τ' and $\tau(K)$ are the scalar curvatures for R' and K , respectively. It is known from [8], that $\|\nabla P\|^2 = 2(\tau - \tau^{**})$. Then (4.2) implies

$$(4.3) \quad \tau = \tau' - \frac{1}{4} \left(\tau(K) - \|\nabla P\|^2 \right).$$

From (3.7) we obtain

$$\rho(K)(y, z) = -g^{ij}g((\nabla_{e_i}P)z, (\nabla_y P)e_j),$$

from where

$$\tau(K) = g^{ij}g^{ks}g((\nabla_{e_i}P)e_s, (\nabla_{e_k}P)e_j).$$

Hence, applying (2.6), we get

$$(4.4) \quad \tau(K) = \frac{1}{2} \|\nabla P\|^2.$$

From (4.3) and (4.4) it follows

$$(4.5) \quad \tau = \tau' + \frac{1}{8} \|\nabla P\|^2.$$

The equalities (4.2), (4.3), (4.4) and (4.5) implies the following

Proposition 4.1. *Let the curvature tensor R' of the P -connection ∇' be a Riemannian P -tensor on $(M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_2$. Then*

$$(4.6) \quad \|\nabla P\|^2 = -8(\tau' - \tau) = \frac{8}{3}(\tau' - \tau^{**}) = 2\tau(K).$$

□

Corollary 4.2. *Let the curvature tensor R' of the P -connection ∇' be a Riemannian P -tensor on $(M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_2$. Then the following assertions are equivalent:*

- 1) (M, P, g) a Riemannian P -manifold;
- 2) $\tau' = \tau$;
- 3) $\tau' = \tau^{**}$;
- 4) $\tau(K) = 0$.

□

Let the considered manifold with a Riemannian P -tensor of curvature R' of the P -connection ∇' in $\mathcal{W}_3 \cap \mathcal{L}_2$ be 4-dimensional. Since H is a Riemannian P -tensor, then according to [11], we have

$$(4.7) \quad H = \nu(H)(\pi_1 - \pi_2) + \nu^*(H)\pi_3,$$

where $\nu(H) = \frac{\tau(H)}{8}$, $\nu^*(H) = \frac{\tau^*(H)}{8}$, $\tau(H)$ and $\tau^*(H)$ are the scalar curvature of H and its associated one, and

$$\begin{aligned} \pi_1(x, y, z, w) &= g(y, z)g(x, w) - g(x, z)g(y, w), \\ \pi_2(x, y, z, w) &= g(y, Pz)g(x, Pw) - g(x, Pz)g(y, Pw), \\ \pi_3(x, y, z, w) &= g(y, z)g(x, Pw) - g(x, z)g(y, Pw), \\ &\quad + g(y, Pz)g(x, w) - g(x, Pz)g(y, w). \end{aligned}$$

From (3.6) and (3.10) we obtain

$$(4.8) \quad \tau(H) = \frac{4\tau' - \tau(K)}{2}, \quad \tau^*(H) = \frac{4\tau'^* - \tau^*(K)}{2},$$

where τ'^* and $\tau^*(K)$ are the associated scalar curvature to τ' and $\tau(K)$, respectively.

We apply (4.8) to (4.7) and thus we obtain the following

Proposition 4.3. *Let the curvature tensor R' of the P -connection ∇' be a Riemannian P -tensor on a 4-dimensional manifold $(M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_2$. Then*

$$H = \frac{4\tau' - \tau(K)}{16}(\pi_1 - \pi_2) + \frac{4\tau'^* - \tau^*(K)}{16}\pi_3.$$

□

Let \mathcal{L}_1 is the subclass of \mathcal{L}_2 determined by

$$(4.9) \quad R(x, y, Pz, Pw) = R(x, y, z, w).$$

The equalities (4.9) and (3.6) imply the following

Proposition 4.4. *Let $(M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_1$. Then*

$$R = R' - \frac{1}{4}K.$$

□

Corollary 4.5. *Let $(M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_1$. Then*

$$\tau = \tau' - \frac{1}{4}\tau(K), \quad \tau^* = \tau'^* - \frac{1}{4}\tau^*(K).$$

□

Corollary 4.6. *Let $(M, P, g) \in \mathcal{W}_3 \cap \mathcal{L}_1$ and $\dim M = 4$. Then*

$$\tau = \frac{1}{2}\tau(H), \quad \tau^* = \frac{1}{2}\tau^*(H).$$

□

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